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Studies in Nonlinear Stochastic Processes. IV. A Comparison of Statistical Linearization, Diagrammatic Expansion, and Projection Operator Methods

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We compare the methods of statistical linearization, perturbation expansions, and projection operators for the approximate solution of nonlinear multimode stochastic equations. The model equations we choose for this comparison are coupled, nonlinear, first-order, one-dimensional complex mode rate equations. We show that the method of statistical linearization is completely equivalent to the neglect of certain well-defined diagrams in the perturbation expansion resulting in the first Kraichnan–Wyld approximation, and to the retention of only Markovian terms in the projection operator method, i.e., those terms that are local in time.

KEY WORDS: Stochastic processes; nonlinear processes; statistical linearization; perturbation expansions; projection operators.

1. INTRODUCTION

Nonlinear stochastic equations are frequently used to model nonlinear physical systems. The technical difficulties associated with the mathematical and physical analyses of nonlinear problems are well known from studies in turbulence,[3] analytical dynamics,[(2)] statistical mechanics,[(3)] and many other areas of physics and applied mathematics. To circumvent these difficulties in practical calculations, one often specifies a procedure for replacing the nonlinear with a linear system with the hope that the characteristic behavior of the nonlinear system can be maintained with this replacement. In this paper we investigate a number of such linear approximations and the relations among them.

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[3] For a recent review of these techniques see, e.g., Leslie.[(1)]

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When nonlinear phenomena are represented by linear equations, certain features of the interactions are always lost. This loss may or may not be important, depending upon the system and, more specifically, on the properties under investigation. Thus, for instance, the possible appearance of soliton solutions due to coherent effects of nonlinear interactions can never be established with a linearized theory.[4] On the other hand, the effects of the nonlinearities on the frequencies (of oscillatory systems), relaxation times, autocorrelation functions, and spectral densities can often be calculated, to a good approximation, by linearized theories. Of particular interest to us, in continuation of some of our earlier work,[5,6] is the method of statistical linearization pioneered by Caughey[7a] and Crandall.[7b]

The method of statistical linearization[5–7] is a prescription for truncating the dynamic equations of nonlinear systems in a fluctuating environment. The procedure is to replace nonlinear functions by "statistically equivalent" linear functions in such a way that the mean square error due to the replacement be a minimum. Because of the success of this simple method in the calculation of the effects of nonlinearities in certain nonlinear systems,[6–8] a critical examination of its physical and mathematical content is of great interest. To carry out such an examination, we consider a specific stochastic nonlinear multimode model and compare the linearized equations obtained by using statistical linearization with perturbation expansions obtained via diagrammatic analysis and with linearized equations obtained from a projection operator technique. This comparison shows that the method of statistical linearization is equivalent to the "first Kraichnan–Wyld approximation" of perturbation theory, which is obtained by neglecting certain well-defined diagrams in the diagrammatic expansion. The results obtained from statistical linearization are also equivalent to those obtained using a projection operator technique if non-Markovian and some mode-mixing terms are neglected. This demonstration of equivalence places the method of statistical linearization on a more familiar and systematic footing as regards its use in problems of statistical physics. A formal analysis of the connection between these various methods of treating nonlinear equations will be presented elsewhere.[8]

2. THE MODEL

The nonlinear model we discuss here is that of a classical stochastic wave field. The time evolution of the field is given by a set of one-dimensional mode rate equations,

$$\dot{A}(k, t) + (i\omega_k + \gamma_k)A(k, t) + F(A; k) = f_k(t) \qquad (1)$$

Here k denotes a mode index or wavenumber that can take on a continuous and/or discrete set of values determined by the symmetry of the system. We will call a mode of wavenumber k the "kth mode." $A(k, t)$ is the time-dependent complex amplitude of the kth mode; it is just the Fourier coefficient of the kth mode in a Fourier expansion of the field variable. For example, if $\mathscr{A}(x, t)$ is the field variable of interest in configuration space, then

$$\mathscr{A}(x, t) = \sum_k A(k, t) \exp(ikx) \tag{2}$$

The function $F(A; k)$ in Eq. (1) represents nonlinear interactions in the wave field. The function $f_k(t)$ is a stochastic term which fluctuates in time. It represents the effect of random perturbations on the mode amplitude $A(k, t)$. The coefficients ω_k and γ_k are the linear frequency and linear decay rate of the kth mode, respectively.

In Eq. (2) and in the rest of this paper we assume that the wave spectrum is discrete. The continuum case can be constructed by simply replacing sums on k's by integrals. It should be noted that the choice of modes for a mode description of the system is not unique. Indeed, any complete set of basis functions can be used to represent the field variable of interest and the resulting mode rate equations would appear different in each of these representations. The basis set is often chosen to exploit any symmetry properties that may exist in the configuration-space equation of evolution. There is, of course, a corresponding change in the interpretation of the index k with changes in the choice of basis functions.

Nonlinearities that are products of field variables and field variable derivatives become convolutions when expressed in terms of mode amplitudes. For example, a quadratic term $\mathscr{A}^2(x, t)$ when transformed according to Eq. (2) yields the nonlinear function

$$F(A; k) = \sum_{k_1, k_2} \delta(k - k_1 - k_2) A(k_1, t) A(k_2, t) \tag{3}$$

where the Kronecker delta is replaced by a delta function in the continuum case. For the most general quadratic nonlinearity without time derivatives,

$$\frac{\partial^r}{\partial x^r}\left(\frac{\partial^s \mathscr{A}}{\partial x^s}\frac{\partial^p \mathscr{A}}{\partial x^p}\right)$$

the nonlinear function $F(A; k)$ is

$$F(A; k) = \sum_{k_1, k_2} \delta(k - k_1 - k_2) i^{r+s+p}(k_1 + k_2)^r k_1{}^s k_2{}^p A(k_1, t) A(k_2, t) \tag{4}$$

where r, s, and p are integers.

In Eq. (1) both the nonlinear term $F(A; k)$ and the fluctuating term $f_k(t)$ model nonlinear interactions in the physical system represented by this set of equations. To see this, consider a linear deterministic wave field, i.e., one described by the set of mode rate equations

$$\dot{A}(k, t) + (i\omega_k + \gamma_k) A(k, t) = 0 \tag{5}$$

The characteristic times of oscillation and damping of the kth mode in this linear wave field are $1/\omega_k$ and $1/\gamma_k$, respectively. The introduction of nonlinear interactions may cause qualitative as well as quantitative changes in the evolution of the system.[9] In one range of interaction parameters the behavior of the system may be deterministic while another range of parameters may cause essentially random behavior. For example, the nonlinear interactions in a fluid change the character of the flow from laminar to turbulent as one increases the Reynolds number of the flow.[1,10] Nonlinearities may therefore introduce randomness into the wave field. In (1) we have assumed that the nonlinearities can be separated into two different parts represented by $F(A; k)$ and $f_k(t)$. This partition arises from the comparison of time scales of variations in the mode amplitudes induced by the nonlinear interactions with the characteristic times of oscillation ($1/\omega_k$) and damping ($1/\gamma_k$) in the linear wave field. Nonlinearities that induce variations in $A(k, t)$ over a time scale $T_{\mathrm{NL}} \geq 1/\omega_k$, i.e., variations that are slower than or comparable to the oscillations of the linear system, are modeled by the deterministic nonlinear function $F(A; k)$. If $T_{\mathrm{NL}} \gg 1/\omega_k$, then this nonlinear interaction is considered to be weak. Nonlinear interactions that induce variations that are fast compared to the characteristic times of the linear system, i.e., over a time scale T_{ST} such that $T_{\mathrm{ST}} \ll 1/\omega_k$ and $T_{\mathrm{ST}} \ll 1/\gamma_k$, are modeled by the stochastic term $f_k(t)$. The introduction of this stochastic term permits one to consider the effect of these high-frequency changes, i.e., fluctuations, in $A(k, t)$ without having to specify the form of the nonlinear interactions that give rise to them in detail. Finally, any rapid fluctuations induced in the mode amplitudes by externally applied forces are also included in $f_k(t)$.

An example of a system which is represented by an equation of the form (1) is a many-mode laser described by[11]

$$\dot{A}(k, t) + (i\omega_k + \gamma_k)A(k, t) + \sum_{k'} \alpha(k, k')|A(k', t)|^2 A(k, t) = f_k(t) \tag{6}$$

This system has a cubic nonlinearity. The number of photons in the kth mode of the field is given by $n(k, t) = A^*(k, t)A(k, t)$. The evolution equation obtained for $n(k, t)$ from (6) has a quadratic nonlinearity. Another example of a system that leads to coupled nonlinear stochastic mode rate equations is an anharmonic polyatomic molecule immersed in a heat bath (see, e.g., Ref. 12).

The choice

$$F(A; k) = -\frac{1}{2}\gamma_k[A(k, t) + A^*(k, t)] + \frac{i\alpha_k}{8\omega_k}[A(k, t) + A^*(k, t)]^3 \tag{7}$$

where α_k is a real coupling coefficient, leads to the Duffing oscillator equation for mode k provided the fluctuating term $f_k(t)$ is purely imaginary. This is seen by writing the mode amplitude as

$$A(k, t) = A_1(k, t) + (i/\omega_k)A_2(k, t) \tag{8}$$

where $A_1(k, t)$ and $A_2(k, t)$ are real. Using this expression together with (7) in (1) yields

$$\dot{A}_1(k, t) = A_2(k, t) \tag{9}$$

$$\dot{A}_2(k, t) + \gamma_k A_2(k, t) + \omega_k{}^2 A_1(k, t) + \alpha_k A_1{}^3(k, t) = -i\omega_k f_k(t) \tag{10}$$

Combining Eqs. (9) and (10) results in the Duffing oscillator equation for $A_1(k, t)$, i.e.,

$$\ddot{A}_1(k, t) + \gamma_k \dot{A}_1(k, t) + \omega_k{}^2 A_1(k, t) + \alpha_k A_1{}^3(k, t) = -i\omega_k f_k(t) \tag{11}$$

In this paper we consider the specific form of the nonlinearity

$$F(A; k) = \alpha_k \sum_{k_1 k_2 k_3} \delta(k + k_3 - k_1 - k_2) A(k_1, t) A(k_2, t) A^*(k_3, t) \tag{12}$$

where α_k is a complex coupling coefficient which is a measure of the strength of the nonlinearity. Note that only certain forms of nonlinearities in configuration space yield a coupling coefficient α_k that depends only on the single wavenumber k. We choose this form of coupling coefficient for convenience to avoid the proliferation of indices. More complicated nonlinearities can be handled by straightforward extensions of the procedures that we illustrate via our specific choice (12).

Throughout this paper we assume that the fluctuating term $f_k(t)$ in Eq. (1) has a Gaussian distribution with zero mean and second moments given by

$$\begin{aligned} \langle f_k(t) f_{k'}^*(t') \rangle &= D_k\, \delta(k - k')\, \delta(t - t') \\ \langle f_k(t) f_{k'}(t') \rangle &= \langle f_k^*(t) f_{k'}^*(t') \rangle = 0 \end{aligned} \tag{13}$$

where the coefficient D_k is a measure of the strength of the fluctuations of the kth mode. We are thus taking the noise to be delta-correlated in time, in agreement with the earlier statement that $f_k(t)$ models interactions that cause very rapid random variations of the mode amplitudes. We also assume that there are no correlations between fluctuations of different modes. These assumptions imply that in configuration space

$$\langle f(x, t) f^*(x', t') \rangle = D(x - x')\, \delta(t - t') \tag{14}$$

where $f(x, t)$ and $D(x - x')$ are inverse Fourier transforms of the quantities in Eq. (13). Equation (14) is a statement of the translational invariance of the fluctuations, i.e., the spatial correlations of fluctuations depend only on the distance $(x - x')$. It should be noted that the simplifying assumptions of Eq. (13) and hence (14) may not be strictly valid for highly nonlinear systems.

3. SOLUTION OF THE LINEAR STOCHASTIC SYSTEM

In the absence of the deterministic nonlinear interaction term $F(A; k)$, the mode amplitudes obey the equation

$$\dot{A}^{(0)}(k, t) + (i\omega_k + \gamma_k)A^{(0)}(k, t) = f_k(t) \tag{15}$$

where the superscript zero on the mode amplitudes denotes the linear system. This is the normal mode equation for the kth mode of a stochastic linear wave field. It has the form of a Langevin equation for a complex variable. Since we assume the driving force $f_k(t)$ to have a delta-correlated Gaussian distribution, we can construct a Fokker–Planck equation for the probability distribution of the $A^{(0)}(k, t)$ to determine the probability that a mode amplitude is in an interval $(A^{(0)}, A^{(0)} + dA^{(0)})$ at time t given its value initially. We denote the realization of $A^{(0)}(k, t)$ by $A^{(0)}(k) \equiv u$ with $A^{(0)}(k, t = 0) = u_0$. We then obtain the Fokker–Planck equation[13]

$$\frac{\partial P(u, t|u_0)}{\partial t} = (\gamma_k + i\omega_k)\frac{\partial}{\partial u}(uP) + (\gamma_k - i\omega_k)\frac{\partial}{\partial u^*}(u^*P) + D_k\frac{\partial^2 P}{\partial u\,\partial u^*} \tag{16}$$

with the initial condition $P(u, 0|u_0) = \delta(u - u_0)$. The solution of Eq. (16) is the Gaussian distribution

$$P(u, t|u_0) = \frac{2\gamma_k}{\pi D_k[1 - \exp(-2\gamma_k t)]} \times \exp -\frac{2\gamma_k|u - u_0\exp[-(\gamma_k + i\omega_k)t]|^2}{D_k[1 - \exp(-2\gamma_k t)]} \tag{17}$$

At times long compared to the relaxation time $(t \gg 1/\gamma_k)$ this distribution approaches the equilibrium distribution

$$P(u, \infty|u_0) = \frac{2\gamma_k}{\pi D_k}\exp\left(-\frac{2\gamma_k}{D_k}|u|^2\right) \tag{18}$$

independent of the initial value u_0.

From these distributions we can obtain both the time-dependent and the equilibrium moments and correlation functions of the mode amplitudes. Thus, separating the real and imaginary parts of the mode amplitude,

$$u = x + iy \tag{19}$$

we find its average value as a function of time to be

$$\langle A^{(0)}(k); t\rangle \equiv \int_{-\infty}^{\infty} dy \int_{-\infty}^{\infty} dx\, uP(u, t|u_0) = u_0\exp(-i\omega_k - \gamma_k)t \tag{20}$$

which vanishes as $t \to \infty$, i.e.,

$$\langle A^{(0)}(k) \rangle_0 = 0 \tag{21}$$

The variance of the mode amplitude as a function of time is

$$\langle |A^{(0)}(k) - \langle A^{(0)}(k) \rangle|^2; t \rangle = (D_k/2\gamma_k)(1 - e^{-2\gamma_k t}) \tag{22}$$

which at equilibrium becomes

$$\langle |A^{(0)}(k) - \langle A^{(0)}(k) \rangle|^2 \rangle_0 = D_k/2\gamma_k \tag{23}$$

The brackets with the zero subscript $\langle \cdots \rangle_0$ indicate the ensemble average with respect to the equilibrium distribution (18) of the linear system.

4. STATISTICAL LINEARIZATION

We now consider the full nonlinear equation (1). The statistical linearization technique[5–7] replaces the nonlinear function $F(A; k)$ with a "statistically equivalent" linear function by requiring that the mean square error due to the replacement be a minimum. The average of $A(k, t)$ at equilibrium is exactly reproduced by this condition and an approximate expression for the equilibrium variance is obtained. Although these first two moments are an incomplete representation of the complete distribution, they should suffice to obtain good approximations to the equilibrium second-order statistics of the system, i.e., to the variances, correlation functions, and spectral densities.

For a large number of physical systems the nonlinear function $F(A; k)$ can be expressed as a polynomial in the mode amplitudes, having quadratic, cubic, and higher order terms. To use the method of statistical linearization we restrict $F(A; k)$ to odd polynomial terms in $A(k, t)$, e.g., cubic as in Eq. (12) or higher order odd powers,[4] and then replace these terms by a linear term in $A(k, t)$ with a single complex coefficient h_k,

$$F(A; k) \to h_k A(k, t) \tag{24}$$

The mean square error Δ due to this replacement is

$$\Delta = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} |F(A; k) - h_k A(k, t)|^2 \, dt \tag{25}$$

where we have shifted the initial conditions to $t = -\infty$. A variation of Δ with respect to h_k such that $\delta\Delta/\delta h_k = 0$ yields

$$h_k = \langle A^*(k, t) F(A; k) \rangle_t / \langle |A(k, t)|^2 \rangle_t \tag{26}$$

where the subscript t on the brackets $\langle \cdots \rangle_t$ indicates a time average as in Eq. (25). The approximate linear equation that replaces (1) is then

$$\dot{A}(k, t) + (i\omega_k^{(l)} + \gamma_k^{(l)}) A(k, t) = f_k(t) \tag{27}$$

[4] Even nonlinearities often lead to instabilities and must thus be handled with special care. We avoid this problem here.

where

$$\omega_k^{(l)} = \omega_k + \operatorname{Im} h_k \tag{28a}$$

$$\gamma_k^{(l)} = \gamma_k + \operatorname{Re} h_k \tag{28b}$$

The superscript (l) denotes "linearized" results.

The evaluation of the frequency shift $\operatorname{Im} h_k$ and the relaxation rate shift $\operatorname{Re} h_k$ from Eq. (26) requires knowledge of the time-dependent solution of Eq. (1) in order to perform the indicated time averages. The reason for searching for an approximation to the nonlinear equation (1), however, is precisely our inability to obtain such an exact solution. To get around this problem in the calculation of h_k it is necessary to replace the time average in Eq. (26) by an *equilibrium* ensemble average. This is a valid procedure if the system is ergodic. To obtain the equilibrium distribution of the system (1) one can transform it to the corresponding Fokker–Planck equation based on our assumption of the Gaussian, delta-correlated, form of the fluctuating term $f_k(t)$, Eq. (13). This Fokker–Planck equation can in some cases be solved exactly for the equilibrium distribution. In those cases where such an analytic solution cannot be effected, one can obtain an approximate equilibrium solution for the calculation of the averages in (26) from the linearized equation (27). The equilibrium distribution of Eq. (27) is, in analogy with the equilibrium solution (18),

$$P^{(l)}(u, \infty | u_0) = \frac{2\gamma_k^{(l)}}{\pi D_k} \exp\left(-\frac{2\gamma_k^{(l)}}{D_k} |u|^2\right) \tag{29}$$

Through $\gamma_k^{(l)}$ it contains the parameter h_k.

When the distribution (29) is used to form the ensemble averages of Eq. (26) one obtains an equation containing $\operatorname{Re} h_k$ on both sides. This self-consistent expression can then be solved for $\operatorname{Re} h_k$ to obtain the approximate relaxation rate shift of Eq. (28b). Now, $\operatorname{Im} h_k$ is simply related to $\operatorname{Re} h_k$ and is used to calculate the approximate frequency shift of Eq. (28a). The Gaussian distribution (29) leads to a simplification of the expression (26) for h_k since the higher moments of $A(k, t)$ can be simply related to the second moment. Thus, for instance, for the cubic nonlinearity (12) we obtain

$$\begin{aligned}
\langle A^*(k)F(A; k)\rangle_l &= \alpha_k \sum_{k_1 k_2 k_3} \delta(k + k_3 - k_1 - k_2) \\
&\quad \times \langle A^*(k)A(k_1)A(k_2)A^*(k_3)\rangle_l \\
&= \alpha_k \sum_{k_1 k_2 k_3} \delta(k + k_3 - k_1 - k_2)[\delta(k - k_1)\, \delta(k_2 - k_3) \\
&\quad + \delta(k - k_2)\, \delta(k_1 - k_3)]\langle |A(k_1)|^2\rangle_l \langle |A(k_2)|^2\rangle_l \\
&= \alpha_k \langle |A(k)|^2\rangle_l \sum_{k_1} \langle |A(k_1)|^2\rangle_l (2 - \delta_{kk_1})
\end{aligned} \tag{30}$$

where the subscript l indicates an ensemble average with respect to the (linearized) equilibrium distribution (29). It can be noted from (30) that certain nonlinear contributions are entirely neglected by this linearization procedure, namely those terms of $F(A; k)$ that do not contain the mode $A(k)$. Thus if we rewrite the cubic nonlinear function (12) as

$$F(A; k) = 2\alpha_k A(k, t) \sum_{k_1} |A(k_1, t)|^2 - \alpha_k |A(k, t)|^2 A(k, t) + \alpha_k \sum_{k_1 k_2 k_3 \neq k}' A(k_1, t) A(k_2, t) A^*(k_3, t) \tag{31}$$

then only the first two terms on the right-hand side of (31) contribute to $\langle A^*(k) F(A; k)\rangle_l$ in (30). The prime on the summation indicates the restriction on wavenumber shown explicitly in (12). This formulation of statistical linearization thus amounts to the neglect of some of the "mode-mixing" terms [the triple sum in Eq. (31)] and to the replacement of $2\alpha_k \sum_{k_1} |A(k_1, t)|^2 - \alpha_k |A(k, t)|^2$ by the complex parameter h_k.[5] The approximate frequency and relaxation rate shifts are now given implicitly by the relations

$$\begin{aligned} \Delta\omega_k^{(l)} &\equiv \operatorname{Im} h_k = \operatorname{Im} \alpha_k \sum_{k_1} \langle |A(k_1)|^2 \rangle_l (2 - \delta_{kk_1}) \\ \Delta\gamma_k^{(l)} &\equiv \operatorname{Re} h_k = \operatorname{Re} \alpha_k \sum_{k_1} \langle |A(k_1)|^2 \rangle_l (2 - \delta_{kk_1}) \end{aligned} \tag{32}$$

Equations (27), (28), and (32) are the statistical linearization approximations to the full nonlinear problem (1) for the particular nonlinearity (12).

The use of (32) with (28a), (28b), and (27) now permits one to obtain various moments and correlation functions of the mode amplitudes $A(k, t)$. In a previous paper[(6)] we have made a detailed analysis of this method for a stochastically driven nonlinear oscillator (the Duffing oscillator) and have shown there that the results obtained via the method of statistical linearization are in good agreement with exact calculations. As shown in that paper and as also pointed out by Crandall,[(7b)] the use of the variance obtained from the *exact* equilibrium solution of the full nonlinear problem in the right-hand sides of (32), which then become explicit rather than implicit relations, leads to a significant improvement in the agreement with exact calculations.

5. PERTURBATION THEORY

Perturbation solutions to nonlinear dynamic systems are notoriously difficult to construct (see, e.g., Ref. 14). In addition, the convergence proper-

[5] If the linearization (24) were replaced by one that includes linear coupling to other modes, i.e., $F(A; k) \to \sum h_{kk'} A(k', t)$, then the triple sum in (31), i.e., the "mode-mixing" terms, could be included in statistical linearization. We intend to pursue this generalization in a subsequent paper.

ties of such perturbation series solutions, once they are constructed, are almost impossible to establish. Only recently have calculational techniques been developed that yield convergent general solutions to some model nonlinear systems.[15] We are not interested here in finding such a general perturbative solution to (1), but rather in determining which terms in the perturbation solution correspond to the linearization results in Section 4 and which do not.

We again adopt the specific form of a cubic nonlinear driving force as given in Eq. (12). The mode rate equations (1) are now written as

$$\dot{A}(k, t) + (i\omega_k + \gamma_k)A(k, t) + \alpha_k \sum_k{}' A(k_1, t)A(k_2, t)A^*(k_3, t) = f_k(t) \quad (33)$$

Fourier-transforming (33) in time according to

$$A(k, \omega) = (2\pi)^{-1} \int_{-\infty}^{\infty} dt\, A(k, t) \exp(-i\omega t) \quad (34)$$

yields

$$Q_k^{-1}(\omega)A(k, \omega) = -\alpha_k \sum_{k,\omega}{}' A(k_1, \omega_1)A(k_2, \omega_2)A^*(k_3, \omega_3) + g_k(\omega) \quad (35)$$

where

$$Q_k^{-1}(\omega) = \gamma_k + i(\omega_k - \omega) \quad (36)$$

and where the "noise" spectrum $g_k(\omega)$ is given by

$$g_k(\omega) = (2\pi)^{-1} \int_{-\infty}^{\infty} dt\, f_k(t) \exp(-i\omega t) \quad (37)$$

The sum in (35) is over k_1, k_2, k_3 and $\omega_1, \omega_2, \omega_3$ and is restricted in both wavenumber and frequency by the conservation conditions

$$k_1 + k_2 = k_3 + k, \qquad \omega_1 + \omega_2 = \omega_3 + \omega \quad (38)$$

As before, the summations in (35) are replaced by integrals for continuous wavenumbers and frequencies.

The nonlinear integral equation that we must solve, therefore, is

$$A(k, \omega) = A^{(0)}(k, \omega) - \alpha_k Q_k(\omega) \sum_{k,\omega}{}' A(k_1, \omega_1)A(k_2, \omega_2)A^*(k_3, \omega_3) \quad (39)$$

where

$$A^{(0)}(k, \omega) = Q_k(\omega) g_k(\omega) \quad (40)$$

is the Fourier transform of the linear mode amplitude defined by Eq. (15). Since $A^{(0)}(k, t)$ and $f_k(t)$ are Gaussian random processes and since the

Fourier transform is a linear operation, $A^{(0)}(k, \omega)$ and $g_k(\omega)$ are also Gaussian. For simplicity we introduce the composite variable $\xi = (k, \omega)$. We define the propagator

$$S(\xi) \equiv (\alpha_k/\alpha)Q(\xi) \tag{41}$$

where α can be interpreted as the average value of α_k. Its actual value need not be known for our purposes. The parameter α will be used as an expansion parameter in the perturbation series and will be taken to be real. Equation (39) can be rewritten as

$$A(\xi) = A^{(0)}(\xi) - \alpha S(\xi) \sum_{\xi}{}' A(\xi_1)A(\xi_2)A^*(\xi_3) \tag{42}$$

Equation (42) is similar in structure to the Duffing oscillator equation studied by Morton and Corrsin (MC).[16] The detailed form of our propagator $S(\xi)$ is different from theirs since they analyze a second-order equation, while (33) is first order. Also, the mode amplitudes here are complex, whereas the oscillator displacements are real. There remains sufficient similarity, however, that the concepts developed in their diagrammatic technique for summing of perturbation series can be applied to the present problem.

Following MC, we expand the mode amplitudes in the series

$$A(\xi) = A^{(0)}(\xi) + A^{(1)}(\xi) + A^{(2)}(\xi) + \cdots \tag{43}$$

where the superscripts indicate the implicit order of α in the terms of the series. By inserting (43) into (42) and equating coefficients of like powers of α, we obtain

$$\begin{aligned}
A^{(0)}(\xi) &= Q(\xi)g(\xi) \\
A^{(1)}(\xi) &= -\alpha S(\xi) \sum_{\xi}{}' A^{(0)}(\xi_1)A^{(0)}(\xi_2)A^{(0)*}(\xi_3) \\
A^{(2)}(\xi) &= -\alpha S(\xi) \sum_{\xi}{}' [2A^{(0)}(\xi_1)A^{(1)}(\xi_2)A^{(0)*}(\xi_3) + A^{(0)}(\xi_1)A^{(0)}(\xi_2)A^{(1)*}(\xi_3)] \\
A^{(3)}(\xi) &= -\alpha S(\xi) \sum_{\xi}{}' \{2[A^{(0)}(\xi_1)A^{(2)}(\xi_2)A^{(0)*}(\xi_3) \\
&\quad + A^{(0)}(\xi_1)A^{(1)}(\xi_2)A^{(1)*}(\xi_3)] + A^{(0)}(\xi_1)A^{(0)}(\xi_2)A^{(2)*}(\xi_3) \\
&\quad + A^{(1)}(\xi_1)A^{(1)}(\xi_2)A^{(0)*}(\xi_3)\} \\
&\;\;\vdots
\end{aligned} \tag{44}$$

At each order in the hierarchy given by (44) one can insert the amplitudes from the preceding order, thereby obtaining expressions solely in terms of $A^{(0)}(\xi)$, the propagator $S(\xi)$, and the coupling coefficient α. When this is done, all the terms in $A^{(n)}(\xi)$ are of $O(\alpha^n)$, as they should be. Inserting (44) into (43) and writing only the first few terms yields

$$\begin{aligned} A(\xi) = {} & A^{(0)}(\xi) - \alpha S(\xi) \sum_{\xi}{}' A^{(0)}(\xi_1) A^{(0)}(\xi_2) A^{(0)*}(\xi_3) \\ & + \alpha S(\xi) \sum_{\xi}{}' \{2A^{(0)}(\xi_1) A^{(0)*}(\xi_3) \alpha S(\xi_2) \sum_{\xi_2}{}' A^{(0)}(\xi_4) A^{(0)}(\xi_5) A^{(0)*}(\xi_6) \\ & + A^{(0)}(\xi_1) A^{(0)}(\xi_2) \alpha S^*(\xi_3) \sum_{\xi_3}{}' A^{(0)*}(\xi_4) A^{(0)*}(\xi_5) A^{(0)}(\xi_6)\} \\ & + \cdots \end{aligned} \tag{45}$$

One can now proceed as Morton and Corrsin and construct a diagrammatic representation of the terms in (45), specify the rules for taking the products in $A(\xi)A^*(\xi)$, and thereby obtain $\langle |A(k)|^2 \rangle$.

We have carried out the indicated perturbation expansion via the usual lengthy and tedious diagrammatic analysis. We found that when one retains only those diagrams that correspond to the so-called "first Kraichnan–Wyld approximation"[(16)] one obtains

$$A^{(r)}(\xi) = \frac{g_k(\omega)}{(\gamma_k + 2M_k \operatorname{Re} \alpha_k) - i(\omega - \omega_k - 2M_k \operatorname{Im} \alpha_k)} \tag{46}$$

for the *renormalized* mode amplitude. The superscript r refers to renormalized quantities. The function M_k appearing in (46) is closely related to the renormalized mean square amplitude and is given by

$$M_k \equiv \tfrac{1}{2} \sum_{k_1} \langle |A^{(r)}(k_1)|^2 \rangle_r (2 - \delta_{kk_1}) \tag{47}$$

The function M_k must be obtained self-consistently using Eq. (46). The subscript r in (47) indicates an equilibrium ensemble average with respect to the renormalized equilibrium distribution. The above expression for $A^{(r)}(\xi)$ can be recognized as the time Fourier transform of the renormalized linear mode rate equation

$$\dot{A}^{(r)}(k, t) + (i\omega_k^{(r)} + \gamma_k^{(r)}) A^{(r)}(k, t) = f_k(t) \tag{48}$$

with

$$\omega_k^{(r)} = \omega_k + 2M_k \operatorname{Im} \alpha_k, \qquad \gamma_k^{(r)} = \gamma_k + 2M_k \operatorname{Re} \alpha_k \tag{49}$$

The frequency and damping coefficients are thus shifted by

$$\Delta\omega_k^{(r)} = 2M_k \operatorname{Im} \alpha_k, \qquad \Delta\gamma_k^{(r)} = 2M_k \operatorname{Re} \alpha_k \tag{50}$$

These results are identical to those obtained by statistical linearization as shown in Eq. (32).[6]

[6] For the specific case of the Duffing oscillator, the equivalence of the "first Kraichnan–Wyld approximation" to the results of statistical linearization has already been pointed out by MC.[(16)]

It should be noted that the "improved" statistical linearization results obtained by calculating M_k using the *exact* equilibrium distribution of the nonlinear problem would correspond to retaining diagrams beyond those of the first Kraichnan–Wyld approximation.

6. PROJECTION OPERATOR METHOD

A formally exact solution of the full mode coupled problem (1) can be obtained with the projection operator method introduced by Zwanzig.[17] We use an extension of the method used by Bixon and Zwanzig[18] in their analysis of the Duffing oscillator and begin by writing the Fokker–Planck equation for the conditional probability distribution $P(\mathbf{u}; t/\mathbf{u}_0)$ of the set of mode amplitudes $\mathbf{u} \equiv \{A(k_i)\}$ at time t given their initial values $\mathbf{u}_0 \equiv \{A(k_i, t = 0)\}$. Rather than dealing with the full nonlinear system (1), we consider the modified set of mode rate equations

$$\dot{A}(k, t) + (i\omega_k + \gamma_k)A(k, t) + F'(A; k) = f_k(t) \tag{51}$$

where $F'(A; k)$ consists of only those terms of $F(A; k)$ that explicitly contain the mode amplitude $A(k, t)$. As seen from Eq. (31), the modified nonlinearity is thus given by

$$F'(A; k) = 2\alpha_k A(k, t) \sum_{k_1} |A(k_1, t)|^2 - \alpha_k |A(k, t)|^2 A(k, t) \tag{52a}$$

and differs from the full nonlinear function $F(A; k)$ by

$$F(A; k) - F'(A; k) = \alpha_k \sum_{k_1 k_2 k_3 \neq k}' A(k_1, t)A(k_2, t)^*(k_3, t) \tag{52b}$$

The terms on the right-hand side of (52b) are precisely the ones that are neglected in statistical linearization, as discussed in Section 4. Since our purpose here is to show how statistical linearization results can be obtained via the projection operator method and since these mode mixing terms introduce some complications in the method, we omit them from the outset.

The conditional probability distribution $P'(\mathbf{u}; t|\mathbf{u}_0)$ for the modified system (51) obeys the Fokker–Planck equation

$$\frac{\partial P'}{\partial t} = \sum_{j=1}^{N} \left\{ \frac{\partial}{\partial u_j} [(i\omega_{k_j} + \gamma_{k_j})u_j + F'(\mathbf{u}; k_j)]P' + \text{c.c.} + D_{k_j} \frac{\partial^2 P'}{\partial u_j \, \partial u_j^*} \right\} \tag{53}$$

where $u_j \equiv A(k_j)$ and c.c. stands for the complex conjugate of all the preceding terms. The equilibrium solution of this Fokker–Planck equation is denoted by $P'_{\rm eq}(\mathbf{u})$. The quantities of particular interest in the following derivation are the average mode amplitudes as a function of time,

$$\langle u_j; t\rangle = \langle A(k_j); t\rangle \equiv \int \cdots \int d\mathbf{u}\, u_j P'(\mathbf{u}; t\,|\,\mathbf{u}_0) \tag{54}$$

where the shorthand notation $d\mathbf{u}$ stands for

$$d\mathbf{u} \equiv \prod_{j=1}^{N} dx_j\, dy_j \tag{55}$$

with

$$u_j \equiv x_j + iy_j \tag{56}$$

We next introduce a projection operator $\mathscr{P}$ whose effect on an arbitrary function $g(\mathbf{u})$ is

$$\mathscr{P}g(\mathbf{u}) = P'_{\rm eq}(\mathbf{u})\left\{\int \cdots \int d\mathbf{u}\, g(\mathbf{u}) + \sum_{j=1}^{N} \langle |u_j|^2\rangle_{\rm eq'}^{-1}\left[u_j^* \int \cdots \int d\mathbf{u}\, u_j g(\mathbf{u}) + \text{c.c.}\right]\right\} \tag{57}$$

The subscript eq′ denotes an average with respect to the equilibrium distribution $P'_{\rm eq}(\mathbf{u})$. The projected probability distribution $\mathscr{P}P'$ is obtained from Eq. (57) with $g(\mathbf{u}) = P'(\mathbf{u}; t\,|\,\mathbf{u}_0)$ as

$$\mathscr{P}P'(\mathbf{u}; t\,|\,\mathbf{u}_0) = P'_{\rm eq}(\mathbf{u})\left\{1 + \sum_{j=1}^{N} \langle |u_j|^2\rangle_{\rm eq'}^{-1}[\langle u_j; t\rangle u_j^* + \text{c.c.}]\right\} \tag{58}$$

The motivation for choosing this particular projection operator is that the projected probability distribution $\mathscr{P}P'$ yields the same average mode amplitude $\langle A(k); t\rangle$ as does the full distribution function P'.

To see that this is the case, we rewrite the Fokker–Planck equation (53) as

$$\frac{\partial P'}{\partial t} = \sum_{j=1}^{N}\left\{\frac{\partial}{\partial |u_j|^2}\left[2\gamma_{k_j}|u_j|^2 + 4(\mathrm{Re}\,\alpha_{k_j})|u_j|^2 \sum_{l} |u_l|^2 - 2(\mathrm{Re}\,\alpha_{k_j})|u_j|^4 + D_{k_j}\right]P' + D_{k_j}|u_j|^2 \frac{\partial^2 P'}{\partial(|u_j|^2)^2}\right\} \tag{59}$$

where we have used the explicit form (52a) for $F'(A; k)$. From Eq. (59) we see that the equilibrium solution $P'_{\rm eq}(\mathbf{u})$ depends only on the absolute square mode amplitude $|u_j|^2$, thus giving the relations

$$\langle u_j\rangle_{\rm eq'} = \langle u_j u_i\rangle_{\rm eq'} = 0, \qquad \langle u_j^* u_i\rangle_{\rm eq'} = \delta_{ij}\langle |u_j|^2\rangle_{\rm eq'} \tag{60}$$

By direct substitution it then follows that the projected probability distribution preserves the average mode amplitude:

$$\begin{aligned}\int \cdots \int d\mathbf{u}\, u \mathscr{P} P'(\mathbf{u}; t \,|\, \mathbf{u}_0) &= \langle u \rangle_{\mathrm{eq}'} + \sum_{j=1}^{N} \langle |u_j|^2 \rangle_{\mathrm{eq}'}^{-1} [\langle u_j^* u \rangle_{\mathrm{eq}'} \langle u_j; t \rangle + \langle u_j u \rangle_{\mathrm{eq}'} \langle u_j^*; t \rangle] \\ &= \langle u; t \rangle = \langle A(k); t \rangle \end{aligned} \tag{61}$$

The second equality in (61) is obtained by using Eq. (60).[7]

The kinetic equation satisfied by the projected probability distribution P' is found by standard methods[(18-20)] to be

$$\frac{\partial}{\partial t} \mathscr{P} P' = \mathscr{P} \mathscr{D} \mathscr{P} P' - \int_0^t ds\, K(s) \mathscr{P} P'(t - s) \tag{62}$$

Here $\mathscr{D}$ is the Fokker–Planck operator of Eq. (53) or of Eq. (59) (i.e., $\partial P'/\partial t = \mathscr{D} P'$), which appears in (62) in place of the usual Liouville operator. The kernel $K(s)$ is the operator

$$K(s) = -\mathscr{P} \mathscr{D} (1 - \mathscr{P}) e^{s(1-\mathscr{P})\mathscr{D}} (1 - \mathscr{P}) \mathscr{D} \mathscr{P} \tag{63}$$

The projected part of $\mathscr{D} \mathscr{P} P'$ in (62) is found from (57) to be given by

$$\begin{aligned}\mathscr{P} \mathscr{D} \mathscr{P} P' = P'_{\mathrm{eq}}(\mathbf{u}) \Big\{ & \int \cdots \int d\mathbf{u}\, \mathscr{D} \mathscr{P} P'(\mathbf{u}; t \,|\, \mathbf{u}_0) \\ & + \sum_{j=1}^{N} \langle |u_j|^2 \rangle_{\mathrm{eq}'}^{-1} [u_j^* \int \cdots \int d\mathbf{u}\, u_j \mathscr{D} \mathscr{P} P'(\mathbf{u}; t \,|\, \mathbf{u}_0) + \text{c.c.}] \Big\} \end{aligned} \tag{64}$$

Inserting the explicit form of $\mathscr{D}$ from Eq. (53) into (64) and integrating by parts yields

$$\begin{aligned}\mathscr{P} \mathscr{D} \mathscr{P} P' = P'_{\mathrm{eq}}(\mathbf{u}) \sum_{j=1}^{N} \langle |u_j|^2 \rangle_{\mathrm{eq}'}^{-1} [& -(i\omega_{k_j} + \gamma_{k_j}) \langle u_j; t \rangle u_j^* \\ & - \langle F'(\mathbf{u}; k_j); t \rangle' u_j^* + \text{c.c.}] \end{aligned} \tag{65}$$

where

$$\langle F'(\mathbf{u}; k); t \rangle' \equiv \int \cdots \int d\mathbf{u}\, F'(\mathbf{u}; k) \mathscr{P} P'(\mathbf{u}; t \,|\, \mathbf{u}_0) \tag{66}$$

Finally, to obtain the equations of motion of the mean amplitudes we take

[7] Equation (61) with the projection operator as defined in (57) would not hold if we had retained the full nonlinear function $F(A; k)$. A more complicated projection operator would have to be defined in order for the average mode amplitudes to be preserved.

the partial derivative of (61) with respect to time and use (62), (64), and (60). The result is

$$\begin{aligned}\frac{\partial}{\partial t}\langle A(k); t\rangle &= \frac{\partial}{\partial t}\langle u; t\rangle \\ &= -(i\omega_k + \gamma_k)\langle u; t\rangle - \langle F'(\mathbf{u}; k); t\rangle' \\ &\quad - \int_0^t ds\, K(s)\langle u; t-s\rangle \end{aligned} \tag{67}$$

The second term on the right-hand side of (67) is explicitly found from (66), (68), and (52) to be

$$\begin{aligned}\langle F'(\mathbf{u}; k); t\rangle' &= \left\{2\alpha_k \frac{\langle |u|^2 \sum_j |u_j|^2\rangle_{\mathrm{eq}'}}{\langle |u|^2\rangle_{\mathrm{eq}'}} - \alpha_k \frac{\langle |u|^4\rangle_{\mathrm{eq}'}}{\langle |u|^2\rangle_{\mathrm{eq}'}}\right\}\langle u; t\rangle \\ &\equiv h_k\langle u; t\rangle \end{aligned} \tag{68}$$

Note that the parameter h_k here is in fact the same as the statistical linearization parameter h_k of Eq. (26) with the nonlinearity F replaced by the modified nonlinearity F' and with the time average replaced by the equilibrium ensemble average.[20] The equation of motion for the average mode amplitude then is

$$\frac{\partial}{\partial t}\langle A(k); t\rangle = -(i\omega_k + \gamma_k + h_k)\langle A(k); t\rangle - \int_0^t ds\, K(s)\langle A(k); t-s\rangle \tag{69}$$

From Eq. (69) it is now readily established that the results of statistical linearization are recovered if the non-Markovian memory term is neglected. The resulting approximate kinetic equation is linear, with frequency and relaxation rate given by

$$\omega_k^{(l)} = \omega_k + \operatorname{Im} h_k, \qquad \gamma_k^{(l)} = \gamma_k + \operatorname{Re} h_k \tag{70}$$

exactly as in (28a) and (28b).

The actual evaluation of h_k presents the same difficulties here as in Section 4 because, in general, we cannot obtain the exact equilibrium solution $P'_{\mathrm{eq}}(\mathbf{u})$, which is needed to evaluate the ensemble averages indicated in (68). It is thus again necessary to evaluate h_k by the self-consistent approximate method outlined in Section 4.

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